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Math 6338 HW 4

Exercise 18, pg. 159 (Exercise 22, pg. 152). Let X be a normed vector space.

(a) If $\mathcal{M}$ is a closed subspace and $x \in X \setminus \mathcal{M}$ then $\mathcal{M} + \mathbb{C}x$ is closed [Use Theorem (5.7c).]

(b) Every finite-dimensional subspace of X is closed.

Proof. (a) Let $x \in X \setminus \mathcal{M}$ and $\{z_n = y_n + \lambda_n x\} \subseteq \mathcal{M} + \mathbb{C}x$, $z \in X$ be such that $z_n \to z$. By Theorem (5.7c), there is an $f \in X^\star$ such that $f(z_n) = |\lambda_n|\delta$ where $\delta = d(x, \mathcal{M}) > 0$. Since f is continuous,

$$z_n \to z \Rightarrow f(z_n) \to f(z) \Rightarrow \{\lambda_n\} \text{ Cauchy } \Rightarrow \exists \lambda \in \mathbb{C} \text{ s.t. } \lambda_n \to \lambda.$$

Also,

$$\begin{aligned} \|y_n - y_m\| &\leq \|y_n + \lambda_n x - z\| + \|z - y_m - \lambda_m x\| + \|\lambda_m x - \lambda_n x\| \\ &= \|z_n - z\| + \|z - z_m\| + |\lambda_m - \lambda_n|\|x\| \to 0, \end{aligned}$$

and $\mathcal{M}$ is closed, so there is a $y \in \mathcal{M}$ such that $y_n \to y$. By linearity of the limiting operation and uniqueness of the limit,

$$z_n = y_n + \lambda_n x \to y + \lambda x = z \in \mathcal{M} + \mathbb{C}x,$$

so $\mathcal{M} + \mathbb{C}x$ is closed.

(b) Let $\mathcal{M}_0 = \{0\}$, then $\mathcal{M}_0$ is a closed subspace of X, so the hypothesis is true for all subspaces of dimension 0. Let $k > 0$ and assume the hypothesis has been proven for subspaces of dimension $k-1$. Let V be a subspace of dimension k with basis $b_1, \ldots, b_k$. Then $V = V' + \mathbb{C}b_k$ where $V' = \text{Span}(b_1, \ldots, b_{k-1})$. By hypothesis, V' is a closed subspace, so applying part (a), V is closed. By induction, the hypothesis holds true for all subspaces of finite dimension.

□

Exercise 19, pg. 159 (Exercise 23, pg. 152). Let X be an infinite-dimensional normed vector space.

(a) There is a sequence $\{x_j\}$ in X such that $\|x_j\| = 1$ for all j and $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$. (Construct x_j inductively, using Exercises 11b and 22.)

(b) X is not locally compact.

Proof. (a) Choose $x_1 \in \mathcal{X}$ such that $\|x_1\| = 1$. Then by part (b) of the problem above, $\mathcal{M}_1 \equiv \mathbb{C}x_1$ is closed. By Exercise 11b (Riesz' lemma), there is an $x_2 \in X$ such that $\|x_2\| = 1$ and $\|x_2 + \mathbb{C}x_1\| = d(x_2, \mathcal{M}_1) \geq \frac{1}{2}$; in particular, $\|x_1 - x_2\| \geq \frac{1}{2}$. Let $\mathcal{M}_2 = \mathcal{M}_1 + \mathbb{C}x_2$; by part (a) of the above problem, $\mathcal{M}_2$ is a closed subspace. For each $n > 2 \in \mathbb{N}$, define the subspace $\mathcal{M}_n = \mathcal{M}_{n-1} + \mathbb{C}x_n$ where x_n is chosen, using Riesz' lemma, to have the properties $\|x_n\| = 1$ and $d(x_n, \mathcal{M}_{n-1}) \geq \frac{1}{2}$. By construction, if $k > j$,

$$\mathcal{M}_j \subseteq \mathcal{M}_{k-1} \Rightarrow d(x_k, \mathcal{M}_j) \geq d(x_k, \mathcal{M}_{k-1}) \geq \frac{1}{2},$$

so in particular, $\|x_k - x_j\| \geq \frac{1}{2}$. Therefore $\{x_j\}$ is a sequence from the unit ball in X satisfying $k \neq j \Rightarrow \|x_k - x_j\| \geq \frac{1}{2}$.

(b) Assume X is locally compact, then there is a compact neighborhood K of 0, so there is an $\epsilon > 0$ such that $B(0; \epsilon) \subset K$. Take a sequence $\{x_j\}$ constructed as in part (a), and define $\{\tau_j = \frac{\epsilon}{2} x_j\}$, then $\|\tau_j\| = \frac{\epsilon}{2}$ and $\|\tau_j - \tau_k\| = \frac{\epsilon}{2}\|x_j - x_k\| \geq \frac{\epsilon}{4}$. Therefore $\{\tau_j\}$ is a net in $B(0; \epsilon) \subset K$ which does not have a convergent subnet, contradicting the compactness of K. This contradiction implies X is not locally compact.

□

Exercise 30, pg. 164 (Exercise 34, pg. 156). Let $\mathcal{Y} = C([0,1])$ and $\mathcal{X} = C^1([0,1])$, both equipped with the uniform norm.

(a) $\mathcal{X}$ is not complete.

(b) The map $D : \mathcal{X} \to \mathcal{Y}$ is closed (see Exercise 7) but not bounded.

Proof. (a) By Lemma (4.47), there is a sequence of polynomials $P_n : [-1,1] \to \mathbb{R}$ such that $||x| - P(x)| \leq \frac{1}{n}$ for all $x \in [-1,1]$. Therefore $P_n \in \mathcal{X} \xrightarrow{u} |\cdot| \notin \mathcal{X}$, showing $\mathcal{X}$ is not complete.

(b) Recall D is closed iff

$$(f_n, Df_n) \to (f, y) \text{ in } \mathcal{X} \times \mathcal{Y} \Rightarrow y = Df.$$

Assume $(f_n, Df_n) \to (f, y)$ in $\mathcal{X} \times \mathcal{Y}$, then $Df_n \to y$ in Y and convergent sequences are bounded in norm, so there is a $B > 0$ so that $|Df_n(x)| \leq \|Df_n\| \leq B$ for all $x \in [0,1]$ and all $n \in \mathbb{N}$, and $B \in L^1([0,1])$. Applying the dominated convergence theorem, $f_n = \int_{[0,1]} Df_n \to \int_{[0,1]} y$; from the uniqueness of the limit, $f_n \to f = \int_{[0,1]} y$, so $y = Df$. Therefore D is closed.

Let $f_n = \frac{\sin(nx)}{n} \in \mathcal{X}$, then $Df_n = \cos(nx)$. Then $\|f_n\| = \frac{1}{n} \to 0$, but $\|Df_n\| = 1$. Therefore $f_n \to 0$ in norm, but $Df_n \not\to 0 = D0$ in norm, so D is not continuous. Therefore D is not bounded.

□

Exercise 38, pg. 164 (Exercise 42, pg. 157). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $\{T_n\}$ be a sequence in $L(\mathcal{X}, \mathcal{Y})$ such that $\lim T_n x$ exists for every $x \in \mathcal{X}$. If $Tx = \lim T_n x$, then $T \in L(\mathcal{X}, \mathcal{Y})$.

Proof. By the linearity of the limiting operation, T is linear. For all $x \in \mathcal{X}$, the sequence $T_n x$ is convergent, so the sequence $\|T_n x\|$ is bounded, and $\sup_{T' \in \{T_n\}} \|T'x\| < \infty$. The Uniform Boundedness Principle then implies $\sup_{T' \in \{T_n\}} \|T'\| = C < \infty$. Therefore, for all $x \in \mathcal{X}$,

$$\|Tx\| = \|\lim T_n x\| = \lim \|T_n x\| \leq \sup_{T' \in \{T_n x\}} \|T'x\| \leq \sup_{T' \in \{T_n\}} \|T'\| \|x\| = C\|x\|,$$

so $\|T\| \leq C < \infty$. Therefore $T \in L(\mathcal{X}, \mathcal{Y})$. □