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Math 6338 HW 2

Exercise 63, pg. 138 (63, pg. 132). . Let $K \in C([0,1] \times [0,1])$. For $f \in C([0,1])$ define $Tf(x) = \int_0^1 K(x,y)f(y)\,dy$. Then $Tf \in C([0,1])$, and $\{Tf : \|f\|_u \leq 1\}$ is precompact in $C([0,1])$.

Proof. Let $\mathcal{F} = \{Tf : f \in C([0,1] \times [0,1]) \text{ and } \|f\|_u \leq 1\}$. Since $[0,1] \times [0,1]$ is compact, $K \in C([0,1] \times [0,1]) = BC([0,1] \times [0,1])$. Therefore there is a $B > 0$ such that $|K(x,y)| \leq B$ for all $(x,y) \in [0,1] \times [0,1]$.

Let $f \in C([0,1])$ be such that $\|f\|_u \leq 1$ and let $x \in [0,1]$, then

$$|Tf(x)| = \left| \int_0^1 K(x,y)f(y)\,dy \right| \leq \int_0^1 |K(x,y)||f(y)|\,dy < B,$$

so Tf is pointwise bounded.

Furthermore, K is continuous on a compact set, so it is uniformly continuous; that is, given any $\epsilon > 0$, there is a $\delta > 0$ such that $d((x_1,y_1),(x_2,y_2)) < \delta \Rightarrow |K(x_1,y_1) - K(x_2,y_2)| < \epsilon$. Notice $d((x,y),(x_1,y)) = |x - x_1|$; this implies that for all $f \in C([0,1])$ such that $\|f\|_u \leq 1$ and all $x \in [0,1]$,

$$|Tf(x) - Tf(x_1)| \leq \int_0^1 |K(x,y) - K(x_1,y)||f(y)|\,dy \leq \int_0^1 |K(x,y) - K(x_1,y)|\,dy < \epsilon$$

when $|x - x_1| < \delta$. Therefore $\mathcal{F}$ is equicontinuous, and $\mathcal{F} \subset C([0,1])$.

Since $[0,1]$ is a compact Hausdorff space, and $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $C([0,1])$, by Ascoli-Arzelà the closure of $\mathcal{F}$ is compact. That is, $\mathcal{F}$ is precompact.

□

Exercise 64, pg. 138 (64, pg. 132). Let (X,ρ) be a metric space. A function $f \in C(X)$ is called **H older continuous of exponent** α ($\alpha > 0$) if the quantity

$$N_\alpha(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} : x \neq y \right\}$$

is finite. If X is compact, $\{f \in C(X) : \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ is compact in $C(X)$.

Proof. Let (X,ρ) be a compact metric space. Fix $\alpha > 0$, define N_α on $C(X)$ as above, and let $\mathcal{A} = \{f \in C(X) : \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$.

Since X is a metric space, it is Hausdorff. Given $\epsilon > 0$, let $\delta = \epsilon^{\frac{1}{\alpha}}$, then for all $f \in \mathcal{A}$ and all $x \in X$,

$$|f(x) - f(y)| \leq N_\alpha(f)\rho(x,y)^\alpha < 1 \cdot \epsilon$$

when $\rho(x,y) < \delta$. Therefore $\mathcal{A}$ is equicontinuous. Also, for all $f \in \mathcal{A}$, $|f| \leq \|f\|_u \leq 1$, so $\mathcal{A}$ is pointwise bounded.

If $f_n \to f$ uniformly with $\{f_n\} \subset \mathcal{A}$, $f \in C(X)$; also,

$$\|f\|_u \leq \|f - f_n\|_u + \|f_n\|_u \leq 1 + \|f - f_n\|_u \to 1 \text{ as } n \to \infty,$$

so $\|f\|_u \leq 1$. Furthermore, for all $x, y \in X$, $x \neq y$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq 2\|f_n - f\|_u + N_\alpha(f_n)\rho(x,y)^\alpha \leq 2\|f_n - f\|_u + \rho(x,y)^\alpha, \end{aligned}$$

and $\|f_n - f\|_u \to 0$ as $n \to \infty$, so

$$|f(x) - f(y)| \leq \rho(x,y)^\alpha \Rightarrow N_\alpha(f) \leq 1.$$

Therefore $\mathcal{A}$ is closed, and by Ascoli-Arzelà, $\overline{\mathcal{A}} = \mathcal{A}$ is compact in $C(X)$. □

Exercise 68, pg. 142 (68, pg. 136). Let X and Y be compact Hausdorff spaces. The algebra generated by functions of the form $f(x, y) = g(x)h(y)$, where $g \in C(X)$ and $h \in C(Y)$, is dense in $C(X \times Y)$.

Proof. Let X and Y be compact Hausdorff spaces. By Proposition 4.10 $X \times Y$ is Hausdorff, and by Tychonoff's Theorem $X \times Y$ is compact. Let A be the algebra generated by functions of the form $f = g \cdot h$, where $g \in C(X)$ and $h \in C(Y)$.

Let $\mathcal{A}$ be the closure of A; by a result shown in class— that the closure of an algebra is an algebra— $\mathcal{A}$ is an algebra in $C(X \times Y)$. If $f \in A$ and $f = g \cdot h$ with $g \in C(X)$, $h \in C(Y)$, then $g^\star \in C(X)$, $h^\star \in C(Y) \Rightarrow f^\star \in A$, so A is closed under complex conjugation. Clearly if $f_n \in A \to f \in \mathcal{A}$, then $f_n^\star \to f^\star \in \mathcal{A}$, so $\mathcal{A}$ is closed under complex conjugation also.

Let (x_1, y_1) and (x_2, y_2) be two distinct points in $X \times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. Since X and Y are compact Hausdorff spaces, they are normal; also $\{x_1\}, \{x_2\}, \{y_1\}, \{y_2\}$ are closed since X and Y are Hausdorff. In the former case, by Urysohn's Lemma there is a $g \in C(X)$ such that $g(x_1) = 1$ and $g(x_2) = 0$ and a $h \in C(Y)$ such that $h(y_1) \neq 0 \neq h(y_2)$ (to form such an h, take the sum of two functions given by Urysohn's lemma); therefore $f = g \cdot h$ separates the points. Similarly, in the latter case, there is an $f \in \mathcal{A}$ which separates the points. Furthermore, by another application of Urysohn's lemma, for any point $(x, y) \in X \times Y$, there is an $f \in \mathcal{A}$ such that $f(x, y) = 1$.

Since $X \times Y$ is a compact Hausdorff space and $\mathcal{A}$ is a closed subalgebra of $C(X \times Y)$ which separates points and is closed under complex conjugation, by the complex Stone-Weierstrass theorem, either $\mathcal{A} = C(X \times Y)$ or $\mathcal{A} = \{f \in C(X \times Y) : f(x_0) = 0\}$ for some $x_0 \in X \times Y$. Since for any point $(x, y) \in X \times Y$, there is an $f \in \mathcal{A}$ such that $f(x, y) = 1$, the latter is true, so A is dense in $C(X \times Y)$.

□