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Math 6337 HW 1

Exercise 1-4. An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e. if $\{E_j\}_1^\infty \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\cup_1^\infty E_j \in \mathcal{A}$).

Proof. Assume $\mathcal{A}$ is a σ-algebra, and let $\cup_1^\infty E_j$ be a countable increasing union of sets $\{E_j\}_1^\infty \subset \mathcal{A}$. Since $\mathcal{A}$ is closed under countable unions, $\cup_1^\infty E_j \in \mathcal{A}$.

Assume $\mathcal{A}$ is an algebra closed under countable increasing unions, and $\{A_i\}_1^\infty \subset \mathcal{A}$. Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^{i} A_j \right),$$

is a countable increasing union of sets in the algebra $\mathcal{A}$, so $\cup_1^\infty A_i \in \mathcal{A}$. Therefore $\mathcal{A}$ is a σ-algebra. □

HW 2

Exercise 1-7. If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

Proof. Assume we have such $\mu_1, \ldots, \mu_n$ and $a_1, \ldots, a_n$, and let $\tilde{\mu} = \sum_1^n a_j \mu_j$. Then

(i) If $E \in \mathcal{M}$, then $\forall i \in \{1, \ldots, n\} : 0 \leq a_i \mu_i(E) \leq \infty$, so $0 \leq \tilde{\mu}(E) \leq \infty$. Furthermore, if $E = \emptyset$, $a_i \mu_i(E) = 0$ for all i, so $\tilde{\mu}(E) = 0$. This shows $\tilde{\mu} : \mathcal{M} \to [0, \infty]$ and $\tilde{\mu}(\emptyset) = 0$.

(ii) Let $\{E_j\}_1^\infty$ be a sequence of disjoint sets in $\mathcal{M}$. Then

$$\tilde{\mu}(\cup_1^\infty E_j) = \sum_{i=1}^{n} a_i \mu_i(\cup_1^\infty E_j) = \sum_{i=1}^{n} a_i \sum_{j=1}^{\infty} \mu_i(E_j) = \sum_{j=1}^{\infty} \sum_{i=1}^{n} a_i \mu_i(E_j) = \sum_{j=1}^{\infty} \tilde{\mu}(E_j).$$

Therefore $\tilde{\mu}$ is a measure on $(X, \mathcal{M})$. □

Exercise 1-9. If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Proof. Taking E, F as above,

$$\mu(E) + \mu(F) = \mu(E \backslash F) + \mu(E \cap F) + \mu(F \backslash E) + \mu(F \cap E)$$

because $E = (E \backslash F) \mathbin{\dot{\cup}} (E \cap F)$ and $F = (F \backslash E) \mathbin{\dot{\cup}} (F \cap E)$. Also, $E \cup F = (E \backslash F) \mathbin{\dot{\cup}} (F \backslash E) \mathbin{\dot{\cup}} (F \cap E)$, so

$$\mu(E \backslash F) + \mu(F \backslash E) + \mu(F \cap E) = \mu(E \cup F),$$

and

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

□

Exercise 1-12. Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$, then $\mu(E) = \mu(F)$.

(b) Say that $E \sim F$ iff $\mu(E \triangle F) = 0$: then $\sim$ is an equivalence relation on $\mathcal{M}$.

(c) If $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \triangle F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$ for all $E, F, G \in \mathcal{M}$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

Proof. Since $(X, \mathcal{M}, \mu)$ is a finite measure space, the difference of the measures of two arbitrary sets in $\mathcal{M}$ is well-defined, and we have:

(a) Since $E \triangle F = (E \backslash F) \dot{\cup} (F \backslash E)$,

$$\mu(E \triangle F) = \mu(E \backslash F) + \mu(F \backslash E) = 0 \Rightarrow \mu(E \backslash F) = \mu(F \backslash E) = 0.$$

Also, $E = (E \backslash F) \dot{\cup} (E \cap F)$ and $F = (F \backslash E) \dot{\cup} (F \cap E)$, so

$$\mu(E) = \mu(E \cap F) = \mu(F).$$

(b) Let $E, F, G \in \mathcal{M}$, then $\sim$ has the following properties:
Reflexivity. $\mu(E \triangle E) = \mu(\emptyset) = 0$, so $E \sim E$.
Symmetricity. $F \triangle E = E \triangle F \Rightarrow \mu(F \triangle E) = \mu(E \triangle F)$, so $F \sim E \Leftrightarrow E \sim F$.
Transitivity. Assume $E \sim F$ and $F \sim G$. By (c) below,

$$0 \leq \mu(E \triangle G) \leq \mu(E \triangle F) + \mu(F \triangle G) = 0,$$

so $\mu(E \triangle G) = 0$ and $E \sim G$. Therefore $\sim$ is an equivalence relation on $\mathcal{M}$.

(c) Let $E, F, G \in \mathcal{M}$, then since $E = (E \backslash F) \dot{\cup} (E \cap F)$,

$$\mu(E \backslash F) = \mu(E) - \mu(E \cap F),$$

and likewise for $\mu(F \backslash E), \mu(G \backslash F), \mu(F \backslash G)$, so

$$\begin{aligned} \rho(E, F) + \rho(F, G) &= \mu(E \backslash F) + \mu(F \backslash E) + \mu(G \backslash F) + \mu(F \backslash G) \\ &= \mu(E) + \mu(G) + 2\mu(F) - 2\left(\mu(F \cap E) + \mu(F \cap G)\right). \end{aligned}$$

From Exercise 1-9, and the facts that $F \cap (E \cup G) \subseteq F$ and $F \cap E \cap G \subseteq E \cap G$,

$$0 \leq 2\left(\mu(E \cap F) + \mu(F \cap G)\right) = 2\left(\mu(F \cap (E \cup G)) + \mu(F \cap E \cap G)\right) \leq 2\mu(F) + 2\mu(E \cap G).$$

Therefore

$$\rho(E, F) + \rho(F, G) \geq \mu(E) + \mu(G) - 2\mu(E \cap G) = \rho(E, G),$$

so ρ satisfies the triangle inequality. Clearly ρ is non-negative, and $\rho(E, F) = 0 \Leftrightarrow E \sim F$. Therefore, ρ is a metric on the set $\mathcal{M}/\sim$.

□

Exercise 1-14. If μ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.

Proof. Let $(X, \mathcal{M}, \mu)$ be a measure space, with μ semifinite, and let $E \in \mathcal{M}$ be such that $\mu(E) = \infty$. Assume there is a $C > 0$ such that for all $F \subset E$, $F \in \mathcal{M}$ satisfying $0 < \mu(F) < \infty$, $\mu(F) \leq C$. Let $\mathcal{F} = \{F \subset E : F \in \mathcal{M} \text{ and } 0 < \mu(F) < \infty\}$; since μ is semifinite, $\mathcal{F} \neq \emptyset$, so $\sup_{F \in \mathcal{F}} \mu(F) = D$ is well-defined and satisfies $0 < D \leq C$. There exists a sequence $\{F_n\}_1^\infty \subseteq \mathcal{F}$ such that $\mu(F_n) \to D$. Consider $\mu(\cup_1^\infty F_n)$—there are two possibilities:

1) $\mu(\cup_1^\infty F_n) = \infty$. In this case, there is an integer k such that $D < \mu(\cup_1^k F_n) < \infty$, contradicting the choice of D.
2) $0 < \mu(\cup_1^\infty F_n) < \infty$. In this case, $\mu(\cup_1^\infty F_n) \leq D$. From $\mu(\cup_{j=1}^i F_j) \geq \mu(F_i)$, it follows

$$\mu(\cup_1^\infty F_n) = \mu(\cup_{i=1}^\infty (\cup_{j=1}^i F_j)) = \lim_{i \to \infty} \cup_{j=1}^i F_j \geq \lim_{i \to \infty} \mu(F_i) = D,$$

so $\mu(\cup_1^\infty F_n) = D$.

Also, $\mu(E\setminus\cup_1^\infty F_n)=\infty$, so there is a $F\subset E\setminus\cup_1^\infty F_n, F\in\mathcal{M}$ such that $0<\mu(F)<\infty$. Then $D<\mu(F\,\dot{\cup}\,(\cup_1^\infty F_n))<\infty$, contradicting the choice of D.

In either case, the choice of D is contradicted. Therefore there is no such integer C. Consequently, for any $C>0$ and any $E\in\mathcal{M}$ satisfying $\mu(E)=\infty$, there is a $F\subset E, F\in\mathcal{M}$ such that $C<\mu(F)<\infty$. $\square$