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Matrix Theory

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HOMEWORK 2

Problem 1. Show that the existence of eigenvalues for matrices in M_n implies the fundamental theorem of algebra.

Proof. Let $q(x) = \sum_{k=0}^{n} \alpha_k x^k$ be an n-th degree polynomial. Let $p = \frac{q}{\alpha_n}$ so $p(x) = 0 \Leftrightarrow q(x) = 0$. The goal is to construct an $A \in M_n$ with characteristic polynomial $p(x)$, so the roots of p are the eigenvalues of A.

Write $p(x) = \sum_{k=0}^{n-1} \beta_k x^k + x^n$; we then know p must annihilate such an A:

$$p(A) = \sum_{k=0}^{n-1} \beta_k A^k + A^n = 0.$$

To ensure that p is the characteristic polynomial, it is sufficient that p also be the minimal polynomial of A, because this would imply that p is the unique monic polynomial of degree n annihilating A. That is, it is sufficient that

$$q(A) = \sum_{k=0}^{m} d_k A^k \neq 0, \quad \forall 0 \neq q \in \mathbf{C}[x] \text{ such that } \deg q = m < n.$$

A sufficient condition for this is that the set $\{A^k e_1\}_{k=0}^{n-1}$ be linearly independent, because this ensures

$$q(A)(e_1) = \sum_{k=0}^{m} (d_k A^k)(e_1) \neq 0. \quad \forall 0 \neq q \in \mathbf{C}[x] \text{ such that } \deg q = m < n.$$

Note that under this condition (that the set $\{A^k e_1\}_{k=0}^{n-1}$ is a basis), if $p(A)(e_1) = 0$, then p annihilates A. To see this, note that any vector $v \in \mathbf{C}^n$ can be written as

$$v = \sum_{k=0}^{n-1} \gamma_k (A^k e_1) = \sum_{k=0}^{n-1} (\gamma_k A^k)(e_1) \equiv q_v(A)(e_1)$$

for some $q_v \in \mathbf{C}[x]$, so

$$p(A)(v) = p(A)(q_v(A)(e_1)) = q_v(A)(p(A)(e_1)) = q_v(A)(0) = 0,$$

so $p(A) = 0$.

Therefore a matrix A with the properties that the set $\{A^k e_1\}_{k=0}^{n-1}$ is linearly independent and $p(A)(e_1) = 0$ has p as its characteristic polynomial. Let

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & -\beta_0 \\ 1 & 0 & \dots & 0 & -\beta_1 \\ 0 & 1 & \dots & 0 & -\beta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\beta_{n-1} \end{bmatrix};$$

clearly the set $\{A^k e_1\}_{k=0}^{n-1}$ is linearly independent, because $A^k e_1 = e_{k+1}$ for $0 \le k \le n-1$. Also,

$$A^n e_1 = \sum_{k=0}^{n-1} -\beta_k A^k e_1 \Rightarrow p(A)(e_1) = 0,$$

so the characteristic polynomial of A is p. By the facts

$$\sigma(A) = \{\lambda : p(\lambda) = 0\} = \{\lambda : q(\lambda) = 0\}$$

and $\sigma(A) \neq 0$, we have the set of roots of q is nonempty.

Clearly every first degree polynomial has a root. Assume every $(n-1)$-th degree polynomial has $n-1$ roots, then let f be an n-th degree polynomial. By the work above, f has a root λ, so

$$f = (x - \lambda)g$$

where g is an $(n-1)$-th degree polynomial. By the inductive assumption, f has n roots. Therefore we have the fundamental theorem of algebra: every n-th degree polynomial has n complex roots.

□

Problem 2. For $A \in M_n$ and $S \in M_n^{-1}$, show $\operatorname{tr} SAS^{-1} = \operatorname{tr} A$.

Let $A = [a_{ij}] \in M_n$, $S = [s_{ij}] \in M_n^{-1}$, and $S^{-1} = [\sigma_{ij}] \in M_n^{-1}$, then

$$\begin{aligned}
\operatorname{tr} SAS^{-1} &= \sum_{i=1}^{n} (SAS^{-1})_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} (SA)_{ik} S^{-1}_{ki} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \sigma_{ki} (SA)_{ik} = \sum_{i=1}^{n}\sum_{k=1}^{n} \sigma_{ki} \left(\sum_{l=1}^{n} S_{il} A_{lk} \right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \sigma_{ki} s_{il} a_{lk} = \sum_{k=1}^{n}\sum_{l=1}^{n} a_{lk} \left(\sum_{i=1}^{n} \sigma_{ki} s_{il} \right) \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} (S^{-1}S)_{kl} a_{lk} = \sum_{k=1}^{n}\sum_{l=1}^{n} I_{kl} a_{lk} \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \delta_l^k a_{lk} = \sum_{k=1}^{n} a_{kk} = \operatorname{tr} A
\end{aligned}$$

so $\operatorname{tr} SAS^{-1} = \operatorname{tr} A$.

Problem 3. Suppose $A_n \in M_n$ is nonsingular. According to (1.1.7), this is equivalent to saying that A has *no* eigenvalues equal to 0. If $\lambda \in \sigma(A)$, show that $\lambda^{-1} \in \sigma(A^{-1})$. If $Ax = \lambda x$ and $x \neq 0$, give an eigenvector of A^{-1} associated with λ^{-1}.

Let $\lambda \in \sigma(A)$, then there is an $0 \neq x \in \mathbb{F}^n$ such that

$$Ax = \lambda x \Rightarrow A^{-1}Ax = \lambda A^{-1}x \Rightarrow x = \lambda A^{-1}x \Rightarrow \lambda^{-1}x = A^{-1}x,$$

so $\lambda^{-1} \in \sigma(A^{-1})$ and x is an eigenvector of A^{-1} associated with λ^{-1}.

Problem 4. Let $A \in M_n(\mathbf{R})$. If λ is a real eigenvalue of A with $Ax = \lambda x$, $0 \neq x \in \mathbf{C}^n$, let $x = \xi + i\mu$, where $\xi, \mu \in \mathbf{R}^n$ are the entrywise real and imaginary parts of x. Show that $A\xi = \lambda\xi$ and $A\mu = \lambda\mu$; conclude that there is a real eigenvector of A associated with λ. Must both ξ and μ be eigenvectors of A? Can there be a real eigenvector associated with a complex non-real eigenvalue of A?

If A and x are as above, then

$$Ax = A(\xi + i\mu) = (A\xi) + i(A\mu) = \lambda x = (\lambda\xi) + i(\lambda\mu) \Rightarrow A\xi = \lambda\xi \text{ and } A\mu = \lambda\mu.$$

Since $x \neq 0$, either $\xi \neq 0$ or $\mu \neq 0$, so there is some $0 \neq \nu \in \mathbf{R}^n$ such that $A\nu = \lambda\nu$; that is, ν is a real eigenvector of A associated with λ. Either ξ or μ may be zero, so in general they are not both eigenvectors of A.

There are no real eigenvector associated with a complex non-real eigenvalue of A: if λ is a complex non-real number and $x \in \mathbf{R}^n$, then $Ax \in \mathbf{R}^n$ but λx is complex and non-real, therefore $Ax \neq \lambda x$.

Problem 5. Consider the block diagonal matrix (0.9.2)

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad A_{ii} \in M_{n_i}$$

Show that the eigenvalues of A are those of A_{11} together with those of A_{22}. *Hint*: First express the eigenvectors of A in terms of those of A_{11} and A_{22}.

Let A be as given, and for $x \in \mathbb{F}^{n_1+n_2}$, let $x^{(u)} = (x_1, \ldots, x_{n_1})$ and $x^{(l)} = (x_{n_1+1}, \ldots, x_{n_2})$. Also, let $0^{(n)} = (\underbrace{0, \ldots, 0}_{n})$.

Assume (λ, x) is an eigenpair for A, then

$$Ax = \lambda x \Rightarrow \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x^{(u)} \\ x^{(l)} \end{bmatrix} = \begin{bmatrix} \lambda x^{(u)} \\ \lambda x^{(l)} \end{bmatrix} \Rightarrow A_{11}x^{(u)} = \lambda x^{(u)} \text{ and } A_{22}x^{(l)} = \lambda x^{(l)}.$$

Since $x \neq 0$, either $x^{(u)} \neq 0$ or $x^{(l)} \neq 0$, so $\lambda \in \sigma(A_{11})$ or $\lambda \in \sigma(A_{22})$; that is,

$$\sigma(A) \subseteq \sigma(A_{11}) \cup \sigma(A_{22}).$$

Assume (λ, x) is an eigenpair for A_{11}, then

$$A \begin{bmatrix} x \\ 0^{(n_2)} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x \\ 0^{(n_2)} \end{bmatrix} = \begin{bmatrix} A_{11}x \\ A_{22}0^{(n_2)} \end{bmatrix} = \begin{bmatrix} \lambda x \\ 0^{(n_2)} \end{bmatrix} = \lambda \begin{bmatrix} x \\ 0^{(n_2)} \end{bmatrix};$$

since $x \neq 0$, $\begin{bmatrix} x \\ 0^{(n_2)} \end{bmatrix} \neq 0$, so $\lambda \in \sigma(A)$. Therefore $\sigma(A_{11}) \subseteq \sigma(A)$.

Likewise, if (λ, y) is an eigenpair for A_{22}, then

$$A \begin{bmatrix} 0^{(n_1)} \\ y \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} 0^{(n_1)} \\ y \end{bmatrix} = \begin{bmatrix} A_{11}0^{(n_1)} \\ A_{22}y \end{bmatrix} = \begin{bmatrix} 0^{(n_1)} \\ \lambda y \end{bmatrix} = \lambda \begin{bmatrix} 0^{(n_1)} \\ y \end{bmatrix};$$

and since $y \neq 0$, $\begin{bmatrix} 0^{(n_1)} \\ y \end{bmatrix} \neq 0$, so $\lambda \in \sigma(A)$. Therefore $\sigma(A_{22}) \subseteq \sigma(A)$.

Therefore $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$.

Problem 6. $A \in M_n$ is called *nilpotent* if $A^q = 0$ for some positive integer q. The minimum such q is called the *index of nilpotence*. Show that all eigenvalues of a nilpotent matrix are 0. In the process, give an example of a nonzero matrix all of whose eigenvalues are equal to 0.

Let $A \in M_n$ be nilpotent with index of nilpotence q. Then by the spectral mapping theorem,

$$\sigma(A^q) = \sigma(0) = \{0\} = x^q(\sigma(A)),$$

so $\lambda \in A \Rightarrow \lambda^q = 0 \Rightarrow \lambda = 0$. Therefore, the nilpotency of A implies $\sigma(A) = \{0\}$.

An example of a nonzero matrix A such that $\sigma(A) = \{0\}$ is the nilpotent matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Problem 7. Find an infinite dimensional vector space and a linear transformation on it with no eigenvalues. Also find a linear transformation T on it such that $T^2 = 0$ but $T \neq 0$.

Let $\mathbb{F}$ be a field, then $\mathbb{F}^{\mathbb{N}}$ is an infinite dimensional vector space, because it is clearly a vector space, and $\left\{\delta_n : k \mapsto \delta_n^k\right\}_{n\in\mathbb{N}}$ is an infinite linearly independent set: if $\alpha_1, \ldots, \alpha_k$ satisfy

$$\begin{aligned} \alpha_1\delta_{n_1} + \cdots + \alpha_k\delta_{n_k} &= 0, \quad \text{then} \\ \alpha_i = (\alpha_1\delta_{n_1} + \cdots + \alpha_k\delta_{n_k})(n_i) &= 0, \quad i = 1, \ldots, k \\ \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k &= 0. \end{aligned}$$

Define $T : \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ by

$$(Tf)(n) = (n \bmod 2) f(n+1),$$

then

$$\begin{aligned} (T(f + \alpha g))(n) &= (n \bmod 2)(f + \alpha g)(n+1) = (n \bmod 2) f(n+1) + (n \bmod 2)(\alpha g)(n+1) \\ &= (Tf)(n) + \alpha(Tg)(n), \end{aligned}$$

so $T(f + \alpha g) = Tf + \alpha Tg$, or T is linear.

Also,

$$T(\delta_6)(5) = (5 \bmod 2)\delta_6(6) = \delta_6(6) = 1,$$

so $T \neq 0$, and

$$T^2(f)(n) = (n \bmod 2)(Tf)(n+1) = (n \bmod 2)((n+1) \bmod 2) f(n+2) = 0,$$

so $T^2 = 0$.

Let $P : \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ be defined by

$$(Pf)(n) = (1 - \delta_1(n)) f(n-1),$$

where $f(0)$ is taken to be 0 for all $f \in \mathbb{F}^{\mathbb{N}}$. Then

$$\begin{aligned} (P(f + \alpha g))(n) &= (1 - \delta_1(n))(f + \alpha g)(n-1) = (1 - \delta_1(n)) f(n-1) + (1 - \delta_1(n))(\alpha g)(n-1) \\ &= (Pf)(n) + \alpha(Pg)(n), \end{aligned}$$

so $P(f+\alpha g) = Pf + \alpha Pg$, or P is linear.

Notice that if $Pf = 0f = 0$, then

$$(Pf)(n+1) = (1-\delta_1(n+1))f(n) = 0 \Rightarrow f(n) = 0, \forall n \in \mathbb{N},$$

so $Pf = 0f$ iff $f = 0$. This shows $0 \notin \sigma(P)$.

Assume $0 \neq \lambda \in \mathbb{F}$ and $f \in \mathbb{F}^{\mathbb{N}}$ satisfy $Pf = \lambda f$. Then

$$Pf = \lambda f \Rightarrow (Pf)(n) = (1-\delta_1(n))f(n-1) = f(n-1) = \lambda f(n), \forall n \neq 1,$$

so by induction $f(n) = \left(\frac{1}{\lambda}\right)^{n-1} f(1)$ for $n > 1$. But

$$(Pf)(1) = 0 = \lambda f(1) \Rightarrow f(1) = 0 \Rightarrow f(n) = 0, \forall n \Rightarrow f = 0.$$

Since there is no nonzero $f \in \mathbb{F}^{\mathbb{N}}$ such that $Pf = \lambda f$, $\lambda \notin \sigma(P)$.

That is, $\sigma(P) = \emptyset$.